THERMALIZATION UNIVERSALITY CLASSES

of weakly nonintegrable many-body systems

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- **terminology and motivation**
- **network classes**
- **measuring thermalization**

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People

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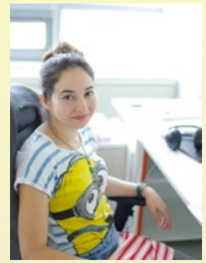
Yagmur Kati



Alexander Cherny

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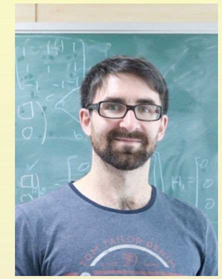

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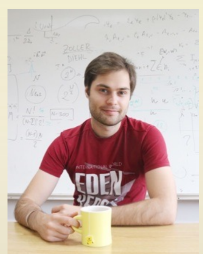
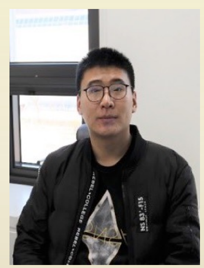
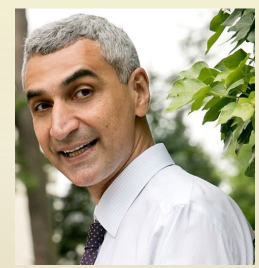


Weihua Zhang



Barbara Dietz

Emil Yuzbashyan



Gabriel Lando

Aniket Patra

Goals

Thermalization dynamics slowing down of many body system in proximity to integrable limit:

- **Use unique action-angle coordinates**
- **Identify different classes of nonintegrable perturbations networks**
- **Quantify thermalization process**
- **Identify novel dynamical regimes**

Measuring Thermalization : measuring time scales !

- **Measure dynamics of observables**
- **Measure and compare their time averages with ensemble averages**
- **Extract time scales**

- **Measure Lyapunov spectra**
- **Invert to obtain Lyapunov times**

Approaching integrable limits

- Time scales will diverge (length scales perhaps as well)
- How will they diverge? How many will diverge? Which ones will diverge?
- Are there different universality classes?
- Can we observe and compute critical exponents?
- Are there further universal quantities?

Nutshell summary

Thermalization of macroscopic systems

Typical systems are nonintegrable

Systems can be tuned close to integrable limits

We studied ergodicity and mixing in ordered and disordered lattices

We found two different classes of weak nonintegrability

LRN: one diverging time scale controls all thermalization dynamics

SRN: one diverging time scale and one diverging (length) scale control the thermalization dynamics

SRN: dramatical slowing down of thermalization

Nutshell summary

Thermalization of macroscopic systems

Typical systems are nonintegrable

Systems can be tuned close to integrable limits

We studied ergodicity and mixing in translationally invariant lattices

We found two different classes of weak nonintegrability

LRN: ordered systems
weak nonlinearity (weak two-body interaction)
all to all nonintegrable interaction between
conserved quantities

SRN: weak finite range lattice coupling/hopping
short range nonintegrable interaction between
conserved quantities
and also
disordered systems and weak nonlinearity (weak two-body int.)

ERGODICITY: Finite time average distribution evolution of action observables

Observables: the actions J(t) of the corresponding integrable limit

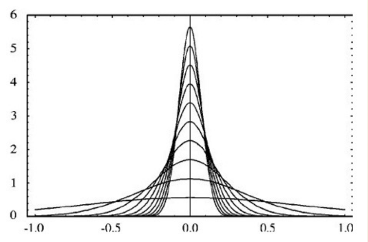
Do their infinite time averages equal ensemble averages?

We don't have infinite time at our disposal!

Finite time averages (FTA)

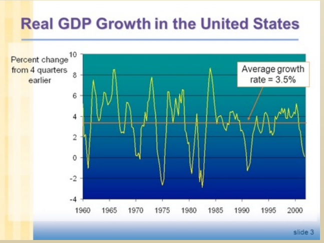
FTA distributions must tend to delta functions for infinite times

Convergence for large averaging times



Extract time scales from that convergence

Note: we do not need to know the ensemble average!



Alternative route: if you know the ensemble average, compute the average of the distance between times when $J(t_n)=\langle J\rangle$

ERGODICITY: Finite time average distribution evolution of action observables

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **95**, 060202(R) (2017)

Intermittent many-body dynamics at equilibrium

C. Danieli,[1,2] D. K. Campbell,[3] and S. Flach[2,1]

PHYSICAL REVIEW LETTERS **122**, 054102 (2019)

Dynamical Glass and Ergodization Times in Classical Josephson Junction Chains

Thudiyangal Mithun,[1] Carlo Danieli,[1] Yagmur Kati,[1,2] and Sergej Flach[1]

PHYSICAL REVIEW E **100**, 032217 (2019)

Dynamical glass in weakly nonintegrable Klein-Gordon chains

Carlo Danieli,[1] Thudiyangal Mithun,[1] Yagmur Kati,[1,2] David K. Campbell,[3] and Sergej Flach[1,4]

PHYSICAL REVIEW E **104**, 014218 (2021)

Fragile many-body ergodicity from action diffusion

Thudiyangal Mithun,[1,2] Carlo Danieli,[3,2] M. V. Fistul,[2,4,5] B. L. Altshuler,[6,2] and Sergej Flach[2,7]

Chaos ARTICLE scitation.org/journal/cha

Thermalization dynamics of macroscopic weakly nonintegrable maps

Cite as: Chaos **32**, 063113 (2022); doi: 10.1063/5.0092032
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View Online Export Citation CrossMark

Merab Malishava[1,2,a)] and Sergej Flach[1,2,a)]

ERGODICITY: Finite time average distribution evolution of action observables

Open Issues

Numerical implementation very demanding

Ambiguity in observable choice

Can be fooled by law of large numbers

Even integrable systems are ergodic, but not mixing!

Lack of aesthetic satisfaction

The Lyapunov Spectrum

Number of Lyapunov exponents = phase space dimension

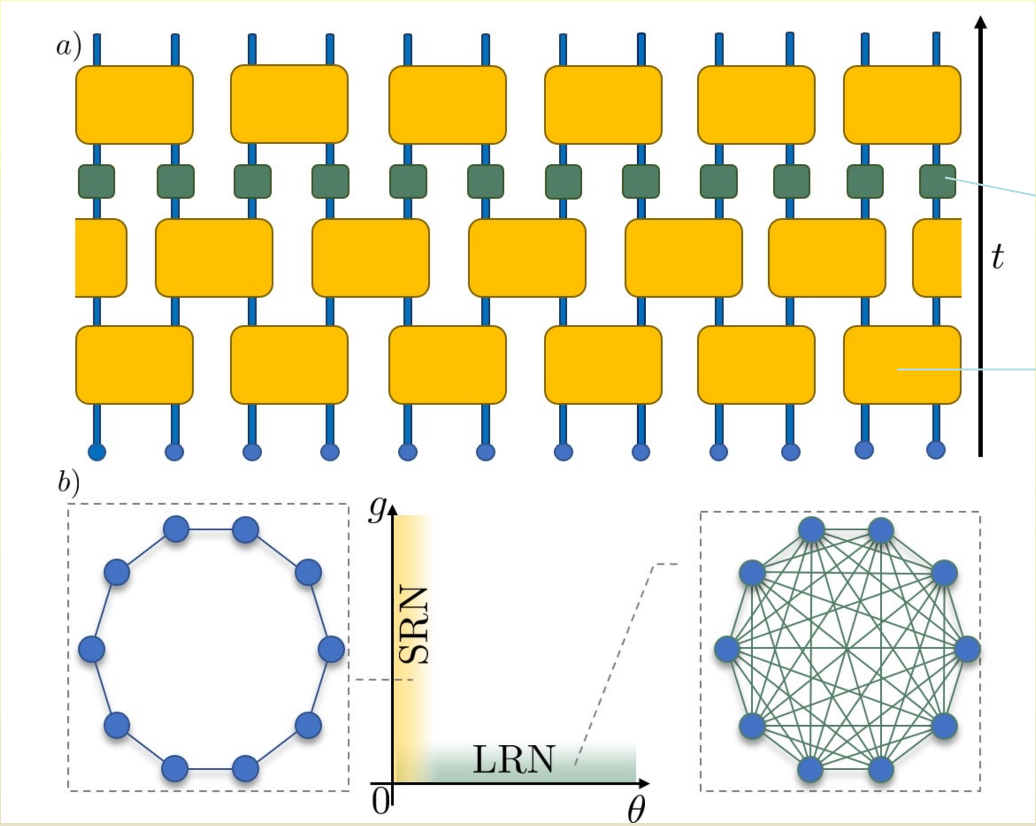
Lyapunov exponents come in $\pm\lambda$ pairs

Per each integral of motion: two zero Lyapunov exponents

TITLE	CITED BY	YEAR
Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1: Theory G Benettin, L Galgani, A Giorgilli, JM Strelcyn Meccanica 15 (1), 9-20	2377	1980

Unitary Circuits for Thermalization

Merab Malishava, SF
PRL 128 134102 (2022)
Chaos 32 063113 (2022)



$$\hat{U} = \left(\prod_{n=1}^{N} \hat{D}_n \hat{G}_n\right)\left(\prod_{n\,\mathrm{odd}} \hat{C}_n\right)\left(\prod_{n\,\mathrm{even}} \hat{C}_n\right)$$

$$\hat{G}_n = e^{ig|\psi_n|^2}|n\rangle\langle n|$$

$$\hat{C}_{n,n+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{\Psi}(t+1) = \hat{U}\vec{\Psi}(t)$$

$$g \to 0 : \ \mathrm{LRN}$$

$$\theta \to 0 : \ \mathrm{SRN}$$

- **Fast numerical evolution due to parallelization**
- **No time discretization roundoff errors (except for roundoff errors)**
- **Versatile, highly efficient unitary map toolbox for long time evolution**

Unitary Circuits for Thermalization

$$\vec{\Psi}(t) = \{\psi_n^A(t), \psi_n^B(t)\}_{n=1}^{N/2}$$

$$\alpha_n^A(t) \equiv \cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t)$$
$$+ \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t)$$

$$\alpha_n^B(t) \equiv \sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t)$$
$$+ \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t) \ .$$

$$\psi_n^A(t+1) = e^{ig|\alpha_n^A|^2}\left[\cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t)\right.$$
$$\left.+ \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t)\right]$$

$$\psi_n^B(t+1) = e^{ig|\alpha_n^B|^2}\left[\sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t)\right.$$
$$\left.+ \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t)\right]$$

Long Range Network (small g)

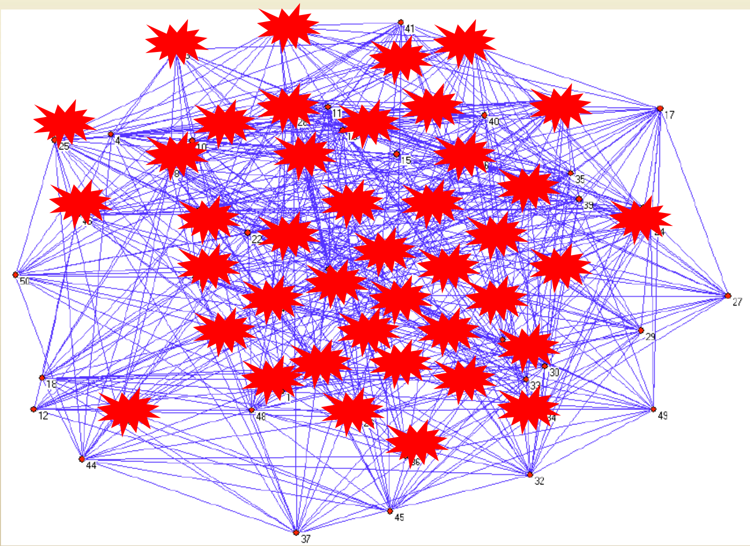
$$\left(\psi_n^A(t), \psi_n^B(t)\right)^T = e^{-i(\omega_k t - kn)} \left(\psi_k^A, \psi_k^B\right)^T$$

$$\vec{\Psi}(t) = \sum_k c_k^r(t) \vec{\Psi}_k^r$$

$$\omega(k) = \pm \arccos\left(\cos^2\theta + \sin^2\theta \cos k\right)$$

$$c_k^r(t+1) = e^{i\omega_k} c_k^r(t) + \frac{ig}{N} \sum_{\substack{r_1,r_2,r_3 \\ k_1,k_2,k_3}} e^{i(\omega_{k_1}^{r_1} + \omega_{k_2}^{r_2} - \omega_{k_3}^{r_3})} I_{k,k_1,k_2,k_3}^{r,r_1,r_2,r_3} c_{k_1}^{r_1}(t) c_{k_2}^{r_2}(t) \left(c_{k_3}^{r_3}(t)\right)^*$$

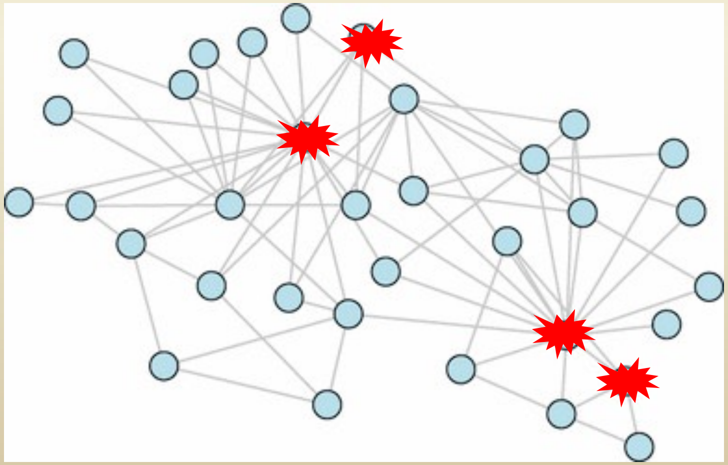
$$I_{k,k_1,k_2,k_3}^{r,r_1,r_2,r_3} = \delta_{k_1+k_2-k_3-k,0} \sum_p \psi_{k_1}^{r_1,p} \psi_{k_2}^{r_2,p} (\psi_{k_3}^{r_3,p})^* (\psi_k^{r,p})^*.$$



Short Range Network (small ϴ)

$$\psi_n^A(t+1) = e^{ig|\alpha_n^A|^2}\left[\psi_n^A(t) - \theta(\psi_{n-1}^B(t) - \psi_n^B(t))\right]$$

$$\psi_n^B(t+1) = e^{ig|\alpha_n^B|^2}\left[\psi_n^B(t) + \theta(\psi_{n+1}^A(t) - \psi_n^A(t))\right].$$



Unitary Circuits for Thermalization

Merab Malishava, SF
PRL 128 134102 (2022)
Chaos 32 063113 (2022)

- **initial state: random local phases, Gibbs-distributed local norm**
- **run the trajectory**
- **add small perturbation, linearize and obtain the tangent map**
- **run 2N trajectories in the tangent map TM**
- **TM1: obtain LLE**
- **TM2: project ⊥ to TM1 and obtain 2nd largest LE**
- **… and so on**

N DoF (sites), $2N$ sorted LEs: $\Lambda_i > \Lambda_j$ with $\{i < j\} \in 1, ..., 2N$

total norm conserved: $\Lambda_N = \Lambda_{N+1} = 0$

unitary evolution: $\Lambda_{i<N} = -\Lambda_{2N-i+1}$

LLE : Λ_1

Irreducible normalized LEs: $\bar{\Lambda}(\rho) = \Lambda_i/\Lambda_1,\ \rho = i/N$

The Lyapunov Spectrum

PRL 128 134102 (2022)
Chaos 32 063113 (2022)
arXiv:2306.14398
arXiv:2308.08921

Fitting the rescaled LS

$$\bar{\lambda}(\rho) = (1 - \rho^{\alpha})\mathrm{e}^{-\beta\rho}$$

Rescaled Kolmogorov-Sinai Entropy Density:
an easy to use class quantifier

$$\kappa = \int_0^1 \bar{\lambda}(\rho)d\rho = \frac{1}{\beta}\left(1 - \mathrm{e}^{-\beta}\right) + \beta^{1-\alpha}\left(\Gamma_{ui}(1+\alpha,\beta) - \Gamma(1+\alpha)\right)$$

(upper incomplete Gamma function)

Unitary Circuits for Thermalization

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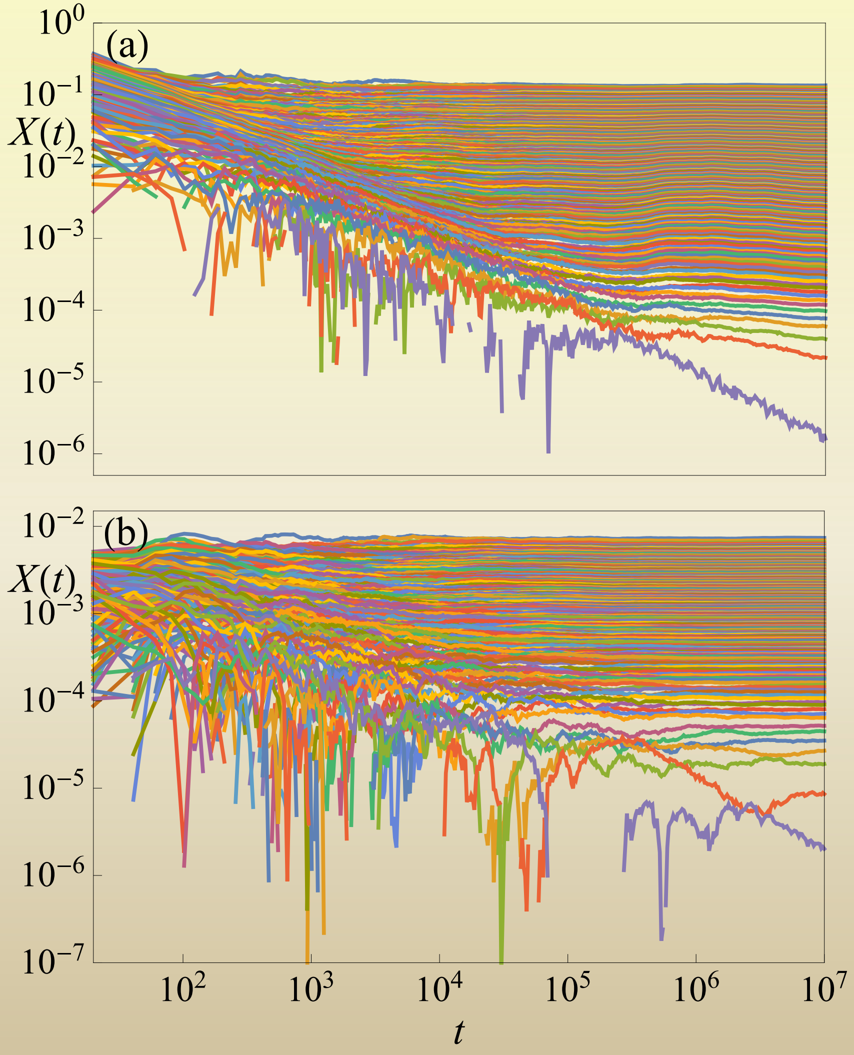
raw data:



Unitary Circuits for Thermalization

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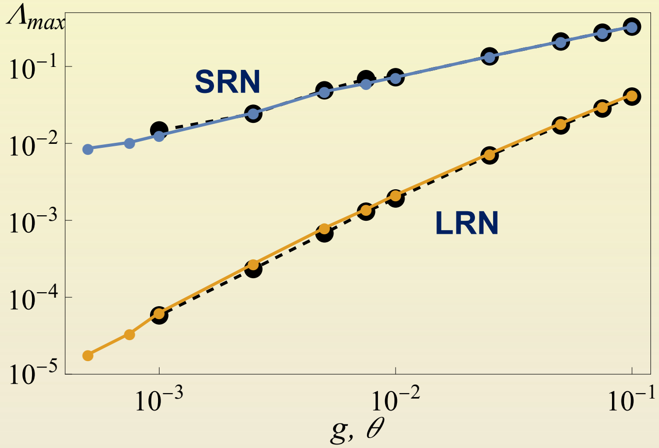
processed data: LLE



integrable limits

Unitary Circuits for Thermalization

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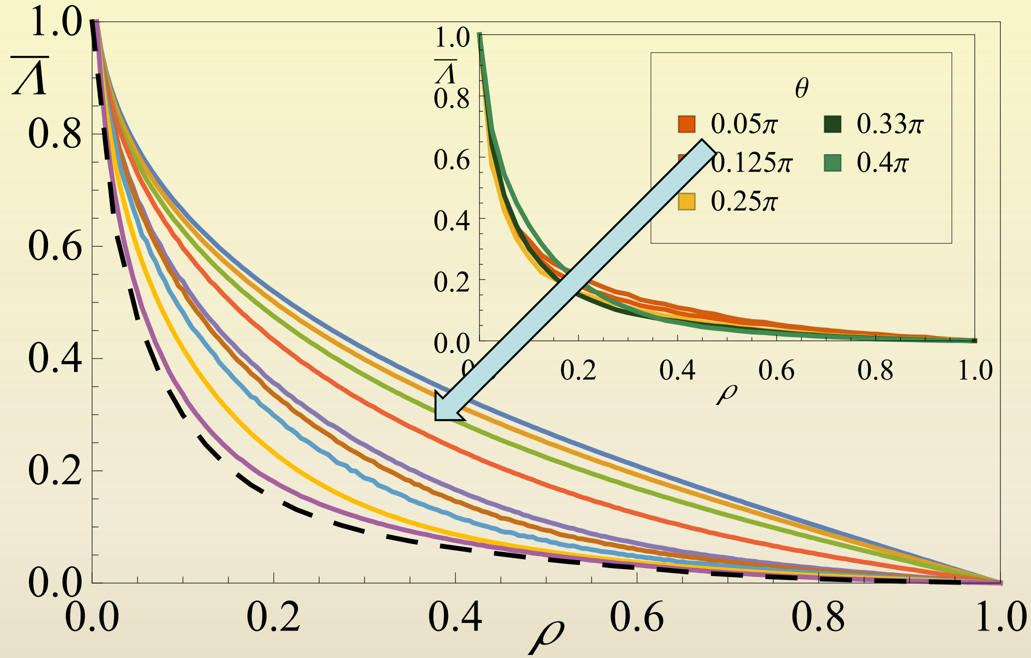
LRN, one diverging scale



$\bar{\Lambda}(\rho)$: analytic function in the LRN limit

$\bar{\lambda}(\rho) = (1 - \rho^{\alpha})\mathrm{e}^{-\beta\rho}$

α, β and κ stay constant nonzero for $g \to 0$

Unitary Circuits for Thermalization

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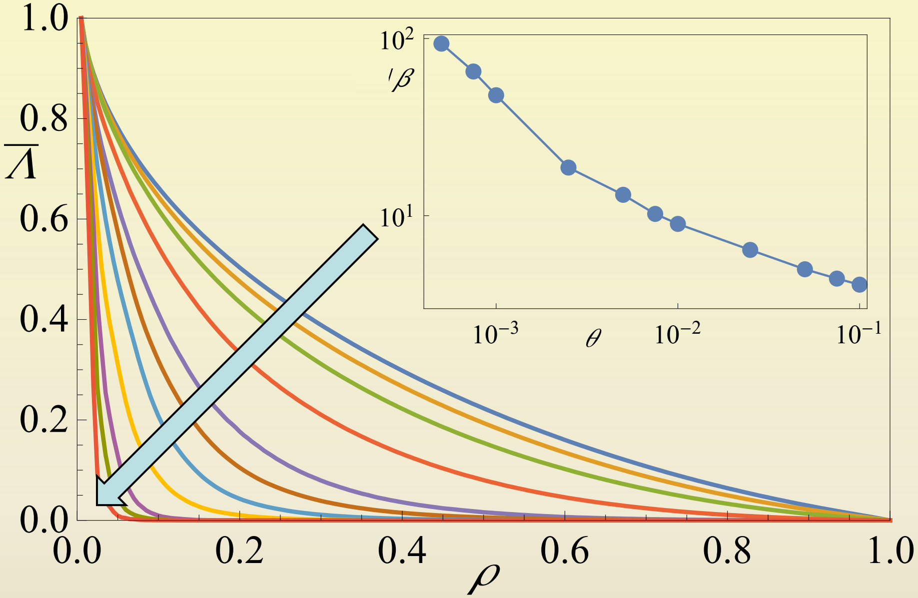
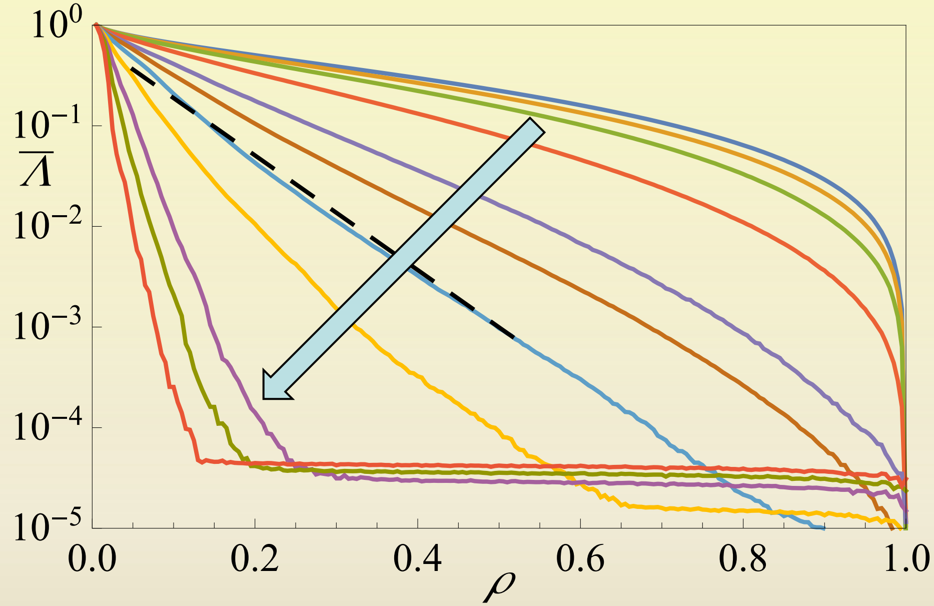
SRN, two diverging scales





$\bar{\Lambda}(\rho)$: non-analytic function in the SRN limit

$$\bar{\lambda}(\rho) = (1 - \rho^{\alpha})\mathrm{e}^{-\beta\rho}$$

β diverges and κ vanishes for $\theta \to 0$

Josephson junction networks in d=1,2,3

Gabriel Lando, SF
arXiv:2306.14398

$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

Number of sites (volume) : N

Energy density: h = H/N

Long Range Network

Gabriel Lando, SF
arXiv:2306.14398

Josephson junction network, energy density h: $h/E_J \ll 1$

$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

$$H_0 = \sum \frac{p_n^2}{2} + \frac{E_J}{2}\left(q_n - q_{n-1}\right)^2 \; : \text{ harmonic chain}$$

$$H_1 = -\frac{E_J}{4} \sum \left(q_n - q_{n-1}\right)^4 \; : \text{ quartic anharmonicity}$$

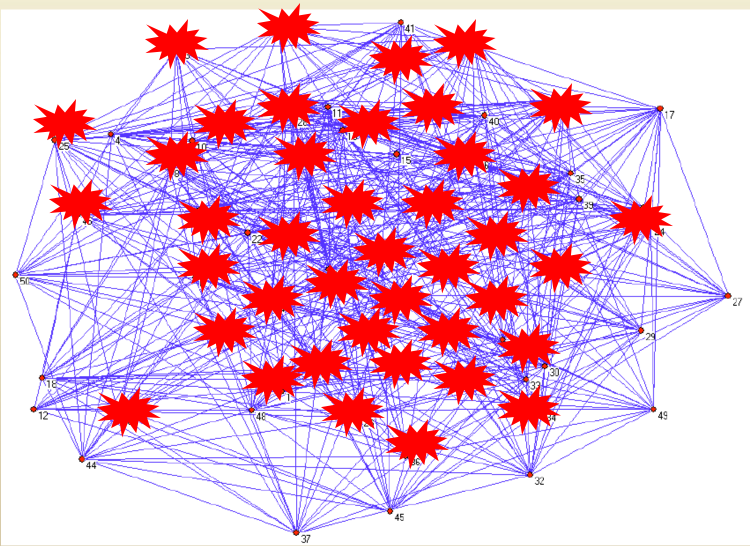
$$L = 3, \; R \sim N^2$$

$\longrightarrow$ Long Range Network

long range network:

$Q_q = \sqrt{2J_q} \sin \Theta_q \; , \; P_q = \omega_q \sqrt{2J_q} \cos \Theta_q$

$\dot{J}_q = -E_J \sum_{q_1,q_2,q_3} \omega_{q_1} \omega_{q_2} \omega_{q_3} A_{q,q_1,q_2,q_3} \sqrt{J_q J_{q_1} J_{q_2} J_{q_3}} cos \Theta_q \sin \Theta_{q_1} \sin \Theta_{q_2} \sin \Theta_{q_3}$



Long Range Network

Gabriel Lando, SF
arXiv:2306.14398

→ **exists due to weak nonlinear all-to-all interactions between normal modes**

→ **nonintegrable nonlinearity local in real space, normal modes extended**

long range network:

$$Q_q = \sqrt{2J_q}\sin\Theta_q\ ,\ P_q = \omega_q\sqrt{2J_q}\cos\Theta_q$$

$$\dot{J}_q = -E_J\sum_{q_1,q_2,q_3}\omega_{q_1}\omega_{q_2}\omega_{q_3}A_{q,q_1,q_2,q_3}\sqrt{J_qJ_{q_1}J_{q_2}J_{q_3}}cos\Theta_q\sin\Theta_{q_1}\sin\Theta_{q_2}\sin\Theta_{q_3}$$

Short Range Network

Gabriel Lando, SF
arXiv:2306.14398

Josephson junction network, $h/E_J \gg 1$

$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

$$H_0 = \sum \frac{p_n^2}{2} \; : \text{ free rotors}$$

$$H_1 = E_J \sum[1 - \cos(q_n - q_{n-1})] \; : \text{ nearest neighbour coupling}$$

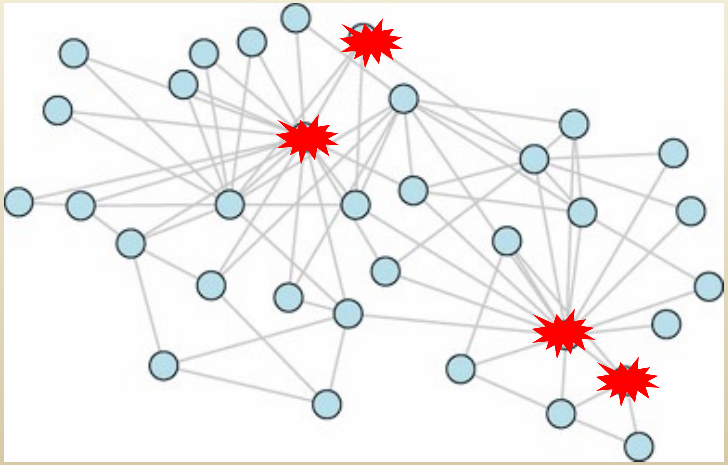
$$L = 1, \; R = 2$$

$\longrightarrow$ Short Range Network

short range network:

$$q_n = \Theta_n \; , \; p_n = J_n$$

$$\dot{J}_n = -E_J \left(\sin(\Theta_n - \Theta_{n-1}) + \sin(\Theta_n - \Theta_{n+1})\right)$$



Short Range Network

Gabriel Lando, SF
arXiv:2306.14398

→ exists due to underlying lattice model structure

→ nonintegrable lattice coupling is local, rotor actions are local

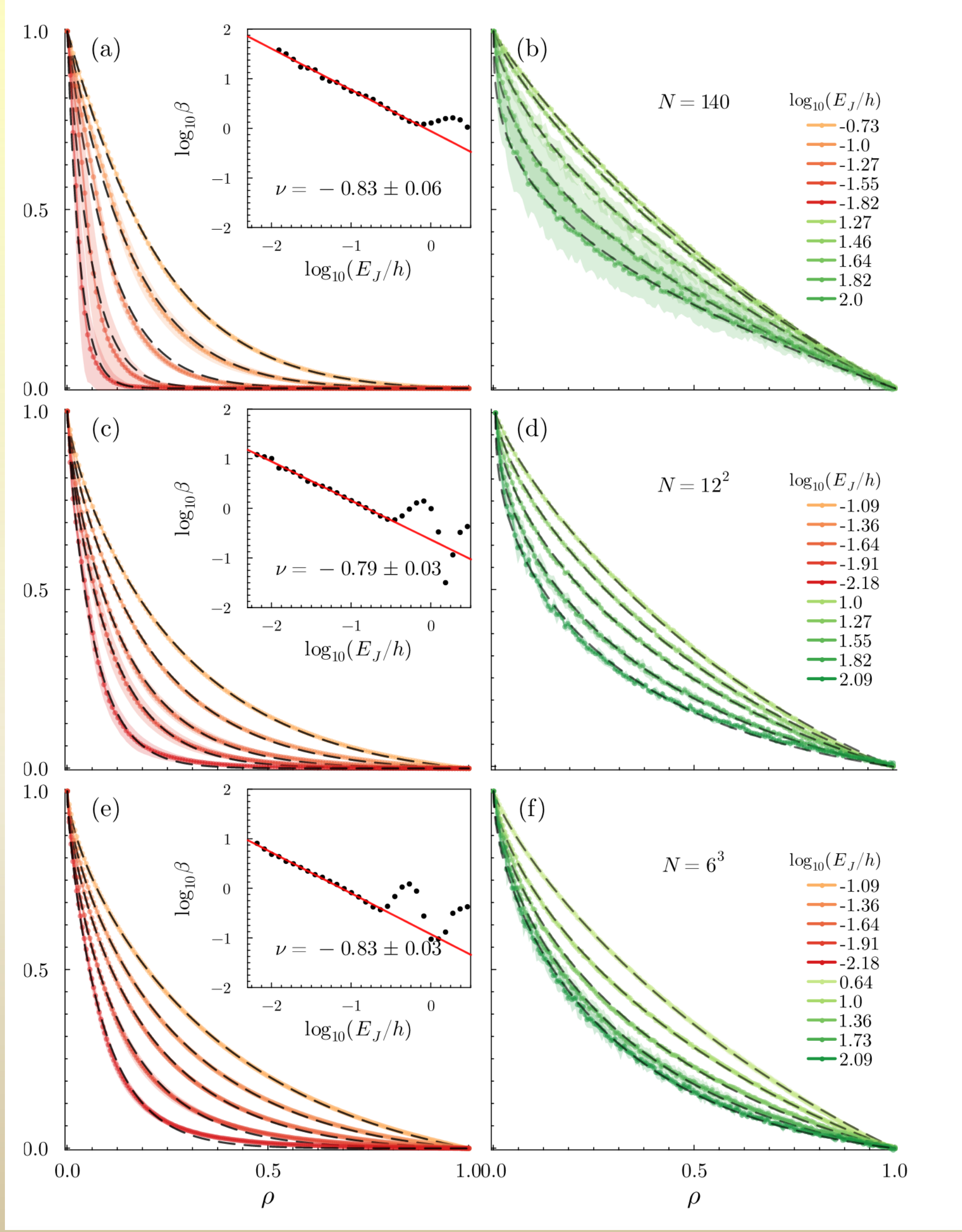
short range network:

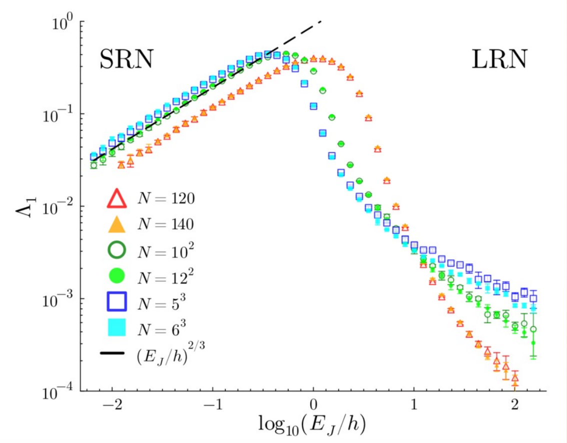
$$q_n = \Theta_n \ , \ p_n = J_n$$

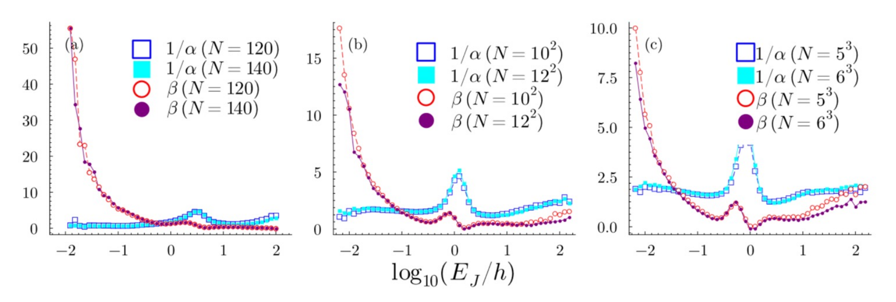
$$\dot{J}_n = -E_J \left(\sin(\Theta_n - \Theta_{n-1}) + \sin(\Theta_n - \Theta_{n+1})\right)$$

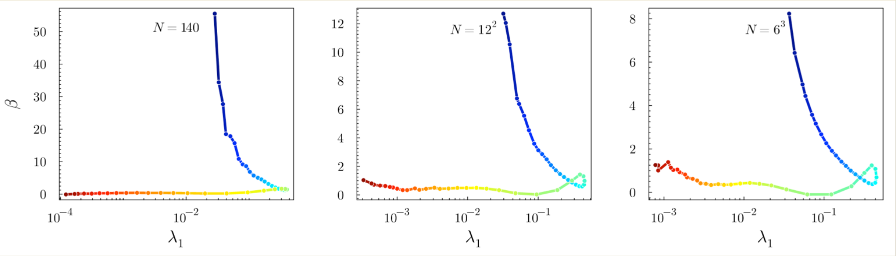
Rescaled Lyapunov spectrum for d=1,2,3

Gabriel Lando, SF
arXiv:2306.14398









Take Home Messages

Merab Malishava, SF
PRL 128 134102 (2022)
Chaos 32 063113 (2022)

Weihua Zhang
Gabriel Lando
Barbara Dietz
SF
arXiv:2306.14398
work in progress

We observe two distinct universality classes

Classifier: nonintegrable perturbation network type

Long range - one LLE controls all thermalization time scales
- renormalized Lyapunov spectrum: analytic function

Short range - one LLE and one (length) scale control the thermalization
- renormalized Lyapunov spectrum: non-analytic function

Disorder and Anderson localization induce transition from LRN to SRN (work in progress)

Explains finite time average observations

Works in any lattice dimension

Outlook

Are there more classes?

Quantitative theories, explanations, predictions?

Other interesting integrable limits: Toda, Ablowitz-Ladik, ideal gas, symplectic integrators for integrable systems

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PHYSICAL REVIEW LETTERS **128,** 134102 (2022)

Lyapunov Spectrum Scaling for Classical Many-Body Dynamics Close to Integrability

Merab Malishava* and Sergej Flach†

arXiv > cond-mat > arXiv:2306.14398

Condensed Matter > Statistical Mechanics

[Submitted on 26 Jun 2023]

Thermalization Slowing-Down in Multidimensional Josephson Junction Networks

Gabriel M. Lando, Sergej Flach

arXiv > nlin > arXiv:2308.08921

Nonlinear Sciences > Chaotic Dynamics

[Submitted on 17 Aug 2023]

Thermalization Universality-Class Transition Induced by Anderson Localization

Weihua Zhang, Gabriel M. Lando, Barbara Dietz, Sergej Flach

Chaos ARTICLE scitation.org/journal/cha

Thermalization dynamics of macroscopic weakly nonintegrable maps

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Merab Malishava[1,2,a] and Sergej Flach[1,2,a]

arXiv > nlin > arXiv:2308.00443

Nonlinear Sciences > Chaotic Dynamics

[Submitted on 1 Aug 2023]

Dynamical chaos in the integrable Toda chain induced by time discretization

Carlo Danieli, Emil A. Yuzbashyan, Boris L. Altshuler, Aniket Patra, Sergej Flach